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# A remark on parameterizing nonsingular cubic surfaces

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## ABSTRACT

Extending a geometric construction due to Sederberg and to Bajaj, Holt, and Netravali, an algorithm is presented for parameterizing a nonsingular cubic surface by polynomials of degree three. The fact that such a parametrization exists is classical. The present algorithm is, by its purely geometric nature, a very natural one. Moreover, it contains a practical way of finding all lines in an implicitly given cubic surface. Two explicit examples are presented, namely the classical Clebsch diagonal surface and the cubic Fermat surface.

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## Introduction

The study of algorithms that parametrize nonsingular real cubic surfaces has increased in recent years due to its many applications outside the area of mathematics.

In particular, it has found application in Computer Aided Geometric Design, where the so-called A-cubic patches (i.e., bounded and nonsingular cubic surfaces in  $\mathbb{R}^3$ ) are used to approximate objects (see Bajaj et al., 1995). Moreover, achieving the lowest possible degree polynomials in the parametrization (which is three for the case of nonsingular cubic surfaces) is critical for the representation and manipulation of these surfaces (see, for example, Schicho, 2006).

The papers Sederberg and Snively (1987) and Bajaj et al. (1998) have treated the subject of parametrization of cubic surfaces in terms of bi-quadratic polynomials using a pair of skew lines on the surface. Since this is the basis of our result, their method is briefly recalled in Section 3 below. We show that a careful choice in one step of their algorithm leads in fact to a parametrization in terms of polynomials of degree three. The paper Berry and Patterson (2001) presents an alternative method using syzygies for parameterizing a cubic surface by polynomials of degree three.

## 1. Preliminaries

A cubic surface  $S \subseteq \mathbb{P}^3$  is defined as the set of zeroes of a homogeneous polynomial  $f$  of degree three in  $\mathbb{P}^3$ , i.e.,

$$S = \{(x : y : z : t) \in \mathbb{P}^3 \mid f(x, y, z, t) = 0\}.$$

Here  $f$  should be irreducible when regarded as a polynomial over  $\mathbb{C}$ . Arthur Cayley and George Salmon discovered that any nonsingular cubic surface over  $\mathbb{C}$  contains precisely 27 lines (see Polo-Blanco, 2007 for a proof following Salmon's reasoning).

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The 27 lines on a cubic surface form a very special configuration. The intersection behaviour of the 27 lines is classically described by the Swiss mathematician Ludwig Schläfli (1858) using the concept of *double six*. A double six is a set of 12 of the 27 lines on a cubic surface  $S$ , represented in Schläfli's notation as:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \end{pmatrix}.$$

The remaining 15 lines are denoted  $b_{i,j}$  with  $1 \leq i < j \leq 6$ . In this notation, two  $a_{i,j}$  intersect if and only if they do not appear in the same row nor in the same column above. A line  $b_{i,j}$  intersects  $a_{k,m}$  if and only if  $m \in \{i, j\}$ . Finally,  $b_{i,j}$  intersects  $b_{k,m}$  if and only if  $\{i, j\}$  and  $\{k, m\}$  are disjoint. Any pair of intersecting lines on  $S$  determines a plane  $P$  containing these lines. The intersection  $S \cap P$  is a union of three lines, and such a plane is called a *tritangent plane*. It is easily verified from the intersection pattern of the lines, that  $S$  contains precisely 45 tritangent planes.

Schläfli was also the first to study cubic surfaces defined over  $\mathbb{R}$ . In Schläfli (1858), he classified the real cubic surfaces according to their number of real lines and real tritangent planes. The same result was also obtained by Cremona (1868). A modern exposition of this may be found in Polo-Blanco and Top (2008).

**Theorem 1.1.** (See Schläfli, 1858 and Cremona, 1868.) *The number of real lines and of real tritangent planes on any nonsingular, real cubic surface is one of the following, in which each pair really occurs: 1. (27, 15), 2. (15, 15), 3. (7, 5), 4. (3, 7) and 5. (3, 13).*

The following lemma will be used in Section 3 below.

**Lemma 1.2.** *Let  $S$  be a nonsingular cubic surface. Given two skew lines  $\ell_1$  and  $\ell_2$  on  $S$ , there exist precisely five lines  $m_1, \dots, m_5$  on  $S$  such that  $\ell_1 \cap m_i \neq \emptyset \neq \ell_2 \cap m_i$  for  $i = 1, \dots, 5$ .*

*Furthermore, all the remaining lines in  $S$  meet at least one of the  $m_i$ .*

**Proof.** This follows using Schläfli's description of the intersection pattern of the 27 lines, as presented above.  $\square$

## 2. Cubic surfaces and Clebsch's result

As noted in the introduction, the cubic algebraic surfaces that are the focus of this paper are defined by an implicit equation  $f(x, y, z, t) = 0$  where  $f$  is a homogeneous polynomial of degree three. An alternative way of representing a cubic surface is in terms of parametric equations whereby  $x, y, z$  and  $t$  are given by homogeneous polynomials  $x(u, v, w)$ ,  $y(u, v, w)$ ,  $z(u, v, w)$ ,  $t(u, v, w)$ .

Alfred Clebsch proved in Beauville (1983) that any nonsingular cubic algebraic surface over the complex numbers can be defined in this way, by choosing the polynomials  $x, y, z, t$  of degree 3 in such a way that they have 6 nontrivial zeroes  $(u, v, w)$  (called *base points*) in common. More precisely, the space of cubic polynomials in  $\mathbb{C}[u, v, w]$  has dimension 10 (note that it is generated by the monomials  $u^3, v^3, w^3, u^2v, u^2w, v^2u, v^2w, w^2u, w^2v, uvw$ ). Consider  $\{p_1, \dots, p_6\} \subseteq \mathbb{P}^2$  six points in general position, which means that no three of them are on a line, and no quadratic curve passes through all six of them. Denote by  $W$  the subspace of cubic polynomials in three homogeneous variables having  $p_1, \dots, p_6$  as zeroes.  $W$  has dimension  $10 - 6 = 4$ . Put  $f_1, \dots, f_4$  a basis of  $W$ . Let

$$\begin{aligned} \Phi: \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\ p &\mapsto (f_1(p) : f_2(p) : f_3(p) : f_4(p)). \end{aligned}$$

**Theorem 2.1.** (See Clebsch, 1871.) *Using the above notation*

- (1)  $\Phi: \mathbb{P}^2 \setminus \{p_1, \dots, p_6\} \rightarrow \mathbb{P}^3$  is a bijection onto its image. The closure of this image is a cubic surface  $S$ .  $\Phi^{-1}$  extends to a map  $\Psi: S \rightarrow \mathbb{P}^2$  and  $\Psi^{-1}(p_i)$  is a line in  $S$  for all  $i = 1, \dots, 6$ .
- (2) Every smooth cubic surface over  $\mathbb{C}$  can be obtained in this way.

In particular, the map  $\Phi$  provides a parametrization of the cubic surface by polynomials of degree 3.

## 3. The algorithm: the complex case

The following algorithm provides a parametrization of a nonsingular cubic surface over  $\mathbb{C}$  by polynomials of degree 3. The construction described in the first two steps of the algorithm is the same as the one used in Sederberg and Snively (1987) and in Bajaj et al. (1998); it can also be found in the proof of Reid (1988, Corollary 7.4). The remain of the algorithm is our contribution. In Step (6) we make a careful choice of a map  $\mathbb{P}^2 \dashrightarrow \ell_1 \times \ell_2$  which is crucial for the achievement of degree three polynomials (instead of bi-quadratic as in Sederberg and Snively, 1987 and in Bajaj et al., 1998) in the total parametrization.

The first step of the algorithm computes a line on the cubic surface. Several authors discuss this problem; compare Sederberg and Snively (1987) and Reid (1988, pp. 103–107). Our algorithm seems, although quite simple, new. It was developed as part of a master's thesis project (Pannekoek, 2009) supervised by one of us. Once any line on the surface is found, the remaining 26 can be easily computed by considering the intersections of  $S$  with planes containing a given line on  $S$ . This was already used in Salmon's classical proof of the existence of the 27 lines. For a detailed exposition, see Reid (1988) or Polo-Blanco (2007).

### 3.1. Finding a line on $S$

In order to find a line on the surface one can proceed as follows. Any line  $\ell \subset \mathbb{P}^3$  is of course determined by two points  $(x_1, x_2, x_3, x_4)$  on it. To find the lines on a given cubic surface with equation  $f(x, y, z, t) = 0$ , first look for lines in the plane given by  $x = 0$ . This means, look for linear factors of  $f(0, y, z, t)$ . A line not in the plane  $x = 0$  contains many points  $(1, x_2, x_3, x_4)$ . Next, for any parameter  $a$ , look for lines in the plane given by  $y = ax$ , i.e., find all  $a$  such that  $f(x, ax, z, t)$  contains a linear factor, and give these factors. Any line not found so far, contains two points  $(1, 0, a, b)$  and  $(0, 1, c, d)$ . Such lines are obtained using the following Maple program.

```
with(PolynomialTools):with(Groebner):
# select a cubic surface
F:=x^3+y^3+z^3+w^3:
# search for lines of the form (1:t:a+ct:b+dt)
rule:={x=1,y=t,z=a+c*t,w=b+d*t}:
F1a:= expand(subs(rule,F)):
F2a:= collect(F1a,t):
vect:=CoefficientVector(F2a,t):
F3a:=[vect[1],vect[2],vect[3],vect[4]]:
G1a:=gbasis(F3a,plex(a,b,c,d)):
sol:=solve({seq(G1a[i],i=1..4)},{a,b,c,d});
```

Note that our algorithm in principle finds all lines. More exactly, the algorithm looks for lines containing two points  $(1, 0, a, b)$  and  $(0, 1, c, d)$ , which means lines having first Plücker coordinate nonzero, or otherwise said, lines not intersecting the line  $L = (0, 0, *, *)$ . It can happen that all real lines in a given cubic surface intersect  $L$ , and in that case our algorithm will only find nonreal lines. Since the choice to look for lines with first Plücker coordinate nonzero is a completely arbitrary one, one may continue with lines having second Plücker coordinate nonzero, so containing points  $(1, a, 0, b)$  and  $(0, c, 1, d)$ . Continuing in this way, we will certainly find (all) real lines.

Notice that Groebner Basis can only be used when working with exact arithmetic. This means that Groebner Basis cannot be computed (by using for example Buchberger algorithm) when the involved polynomials have floating point real numbers as coefficients.

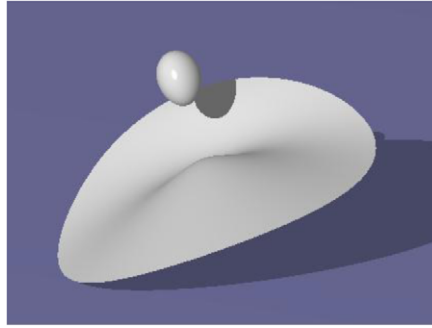
In order to extend our algorithm to finite precision arithmetic (which is the usual case in commercial CAD) one may proceed as follows by using univariate resultants instead of Groebner Basis. According to the structure of the polynomials  $\text{vect}[i]$ , the variables  $a, b$  and  $c$  are eliminated by computing several univariate resultants. First  $a$  is eliminated by computing the resultants  $w_i(b, c, d)$  with respect to  $a$  of  $\text{vect}[i]$  and  $\text{vect}[3]$  with  $i = 0, 1, 2$ . Since  $b$  does not appear in  $w_3$ , then  $b$  is eliminated by computing the resultant  $u(c, d)$  of  $w_1$  and  $w_2$  with respect to  $b$  which is a polynomial depending only on  $c$  and  $d$ . Finally  $c$  is eliminated by computing the resultant  $r(d)$  of  $w_3$  and  $u$  with respect to  $c$  which is an univariate polynomial in  $d$ . By solving the equation  $r(d) = 0$  the values of  $d$  are determined. The values for  $a, b$  and  $c$  can be obtained through a backward analysis of the polynomial system considered or by using subresultants (see Gonzalez-Vega et al., 1994 or Basu et al., 2003). There are several numerical issues to be considered in practice here: our approach suggests to make the computations generically just one time: i.e. the coefficients of the cubic  $F$  are considered as parameters to be specialized to the particular values of the coefficients of the considered cubic surface equation (in a numerically controlled way) in order to get the equation  $r(d) = 0$  and the different subresultants providing the searched values for  $a, b$  and  $c$ .

### 3.2. Sketch of the algorithm

Let  $S$  be a nonsingular cubic surface in  $\mathbb{P}^3(\mathbb{C})$ .

- (1) Find a line on the surface (see Section 3.1).
- (2) Find a line  $\ell_2 \subseteq S$  such that  $\ell_1 \cap \ell_2 = \emptyset$ .
- (3) Define the map  $\Phi_1 : S \subseteq \mathbb{P}^3 \dashrightarrow \ell_1 \times \ell_2$  as follows. Let  $p \in S \setminus (\ell_1 \cup \ell_2)$  be a point on the surface (note that, in fact, the map  $\Phi_1$  is going to be applied to any general point in  $\mathbb{P}^3$ ). Calculate the unique line  $\ell \in \mathbb{P}^3$ , such that  $p \in \ell$ ,  $\ell \cap \ell_1 = \{q_1\}$  and  $\ell \cap \ell_2 = \{q_2\}$ . Define  $\Phi_1(p) = (q_1, q_2) \in \ell_1 \times \ell_2$ .
- (4) Next, take an isomorphism  $f : \ell_1 \times \ell_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  defined as

$$f : ((x_1, x_2, x_3, x_4), (x'_1, x'_2, x'_3, x'_4)) \mapsto ((x_j : x_k), (x'_m : x'_n)).$$



**Fig. 1.** Surface  $x_0^3 + x_1^3 + x_2^3 + x_0^2x_1 + x_0^2x_2 + x_0x_1^2 + x_1^2x_2 - 6x_2^2 + 11x_2 - 6 = 0$  of type 5 in Theorem 1.1.

- (5) Find the five lines  $m_1, \dots, m_5$  on  $S$  that intersect both  $\ell_1$  and  $\ell_2$  (see Lemma 1.2).
- (6) The map  $f \circ \Phi_1 : S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  contracts the five lines  $m_i$  to five points  $p_i = ((x_j(m_i) : x_k(m_i)), (x'_m(m_i) : x'_n(m_i))) \in \mathbb{P}^1 \times \mathbb{P}^1$  for  $i = 1, \dots, 5$  (this is also observed in Reid (1988, Exc. 7.5)). Choose one such image point, say  $p_1$ .
- (7) Consider the space of forms of bi-degree  $(1, 1)$  in  $\mathbb{C}[X_j, X_k, X'_m, X'_n]$  that vanish at the point  $p_1$ . Using a basis  $\{f_1, f_2, f_3\}$  of this space, define a rational map  $\Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$  by  $p \mapsto (f_1(p) : f_2(p) : f_3(p))$ .
- (8) Put  $\Phi := \Phi_2 \circ f \circ \Phi_1 : S \rightarrow \mathbb{P}^2$ . This map has as inverse the desired parametrization.

To define the isomorphism in Step (4) is quite elementary. We present some details here since this will be used in the real case below. Consider  $V = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3 + \mathbb{C}x_4$  in  $\mathbb{C}[x_1, \dots, x_4]$ , and  $W \subseteq V$  the subspace given by  $\{h \in V \mid h(\ell_1) = 0\}$ . The space  $V/W$  has dimension 2. One can consider  $\{x_j, x_k\}$  ( $j \neq k$ ) as a basis of  $V/W$ . Then, the map

$$\ell_1 \rightarrow \mathbb{P}(V/W) \simeq \mathbb{P}^1 : (x_1, x_2, x_3, x_4) \mapsto (x_j : x_k)$$

is an isomorphism. The same can be done for the line  $\ell_2$ , and an analogous basis  $\{x_m, x_n\}$  ( $m \neq n$ ). An isomorphism  $f : \ell_1 \times \ell_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is given by

$$f : ((x_1, x_2, x_3, x_4), (x'_1, x'_2, x'_3, x'_4)) \mapsto ((x_j : x_k), (x'_m : x'_n)).$$

The reason for choosing one of the image points  $p_i$  in Step (6) of the algorithm is the following. We intend to construct a composition of maps

$$S \xrightarrow{f \circ \Phi_1} \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$

such that the composition extends to all of  $S$ , and contracts 6 lines. Of course, these lines have to be pairwise disjoint. Now observe that  $f \circ \Phi_1$  already contracts 5 lines. Lemma 1.2 states that none of the remaining 22 lines on  $S$  is disjoint from all 5 of these. Hence to obtain the desired result, one cannot simply compose  $f \circ \Phi_1$  with a map which is everywhere defined.

The construction described in Step (7) uses the so-called forms of bi-degree  $(1, 1)$  in  $\mathbb{C}[X_j, X_k, X'_m, X'_n]$  (i.e., polynomials that are homogeneous of degree 1 in  $(X_j, X_k)$  and in  $(X'_m, X'_n)$ ). Any such form is given as  $\alpha X_j X'_m + \beta X_j X'_n + \gamma X_k X'_m + \delta X_k X'_n$ , with  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ . Note that the map  $\Phi_2$  contracts the two curves  $p_{1,1} \times \mathbb{P}^1$  and  $\mathbb{P}^1 \times p_{1,2}$ , where  $p_1 = (p_{1,1}, p_{1,2})$ , to two points  $q_1$  and  $q_2$  in  $\mathbb{P}^2$  respectively.

It can be shown (see Polo-Blanco, 2007) that the map  $\Phi$  in Step (8) defines a morphism  $S \rightarrow \mathbb{P}^2$  which contracts 6 lines and is an isomorphism outside these lines.

**Remark 3.1.** The three maps  $\Phi_2$ ,  $f$  and  $\Phi_1$  can be easily inverted. The map  $\Phi^{-1} = \Phi_1^{-1} \circ f^{-1} \circ \Phi_2^{-1}$  provides the parametrization of the surface. It is given by  $p \mapsto (f_1(p) : \dots : f_4(p))$ , where  $\{f_1, \dots, f_4\}$  is a basis of the space of plane cubic curves passing through six points. In particular, it is a map given by cubic polynomials.

**Remark 3.2.** The six points  $\{p_1, \dots, p_6\}$  are obtained as follows: Four of them are  $\Phi(m_2)$ ,  $\Phi(m_3)$ ,  $\Phi(m_4)$ ,  $\Phi(m_5)$ . The remaining two are  $\Phi_2(r_1)$  and  $\Phi_2(r_2)$ , in which  $r_i$  denote the lines  $p_{1,1} \times \mathbb{P}^1$  and  $\mathbb{P}^1 \times p_{1,2}$ , with  $p_1 = (p_{1,1}, p_{1,2}) \in \mathbb{P}^1 \times \mathbb{P}^1$ .

#### 4. Algorithm: the real case

Now we assume that the cubic surface  $S$  is real. It can be shown (see Polo-Blanco and Top, 2008) that all nonsingular, cubic surfaces defined over  $\mathbb{R}$  possess a real, rational parametrization except those that have 3 real lines and 13 real tritangent planes (type 5 in Schläfli's classification, see Fig. 1). The reason why this is true, is that the real locus of a type-5 surface is not connected, and the real projective plane is. Connectedness is a property that is preserved under such a contraction.

We assume  $S$  is not of type 5 in Schläfli's classification, and distinguish the following two cases:

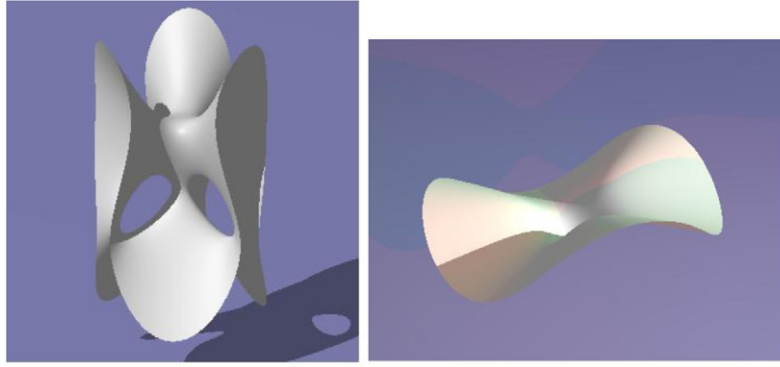


Fig. 2. Clebsch diagonal surface (left) and Fermat surface (right).

- (i)  $S$  contains a skew pair of real lines  $\{\ell_1, \ell_2\}$ . In this case, the algorithm is exactly the same as in the complex case, where one should replace  $\mathbb{C}$  by  $\mathbb{R}$  everywhere.
- (ii) The real locus of  $S$  is connected, but  $S$  does not contain a skew pair of real lines. In this case, there exists a skew pair of complex conjugate lines  $\{\ell, \bar{\ell}\}$ , where  $\bar{\ell}$  denotes the complex conjugate of the line  $\ell$ .

We briefly sketch the steps to follow in this case:

- (1) Choose  $\{\ell, \bar{\ell}\}$ , a pair of skew conjugate lines on  $S$ .
- (2) Build a map  $\Phi_1 : S \subseteq \mathbb{P}^3 \dashrightarrow \ell \times \bar{\ell}$  as in the complex case.
- (3) Define the map  $f : \ell \times \bar{\ell} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  as

$$f : ((x_1, x_2, x_3, x_4), (x'_1, x'_2, x'_3, x'_4)) \mapsto ((x_j : x_k), (x'_j : x'_k)).$$

- (4) Find the five lines  $m_1, \dots, m_5$  on  $S$  that intersect both  $\ell$  and  $\bar{\ell}$ . Consider one real line among them, say  $m_1$  (note that since the set  $\{\ell, \bar{\ell}\}$  is defined over  $\mathbb{R}$ , so is  $\{m_1, \dots, m_5\}$ ).
- (5) The map  $f \circ \Phi_1$  contracts the five lines  $m_i$  to five points  $p_i \in \mathbb{P}^1 \times \mathbb{P}^1$  for  $i = 1, \dots, 5$ . As in the complex case, one chooses one of the image points  $p_i$ . In this case, the chosen point is the real point  $p_1 \in \mathbb{P}^1 \times \mathbb{P}^1$ .
- (6) Define the map  $\Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$  by  $p \mapsto (f_1(p) : \dots : f_3(p))$  where  $\{f_1, f_2, f_3\}$  is a basis of the space of forms of bi-degree  $(1, 1)$  vanishing at  $p_1$  and invariant under complex conjugation.
- (7) The complete real morphism is  $\Phi = \Phi_2 \circ f \circ \Phi_1 : S \rightarrow \mathbb{P}^2_{\mathbb{R}}$ . It contracts 6 lines consisting of 3 pairs of conjugated complex lines.

The map  $f$  in Step (3) is defined as follows. Let  $V_{\mathbb{R}} = \mathbb{R}x_1 + \dots + \mathbb{R}x_4$  and  $V = \mathbb{C}x_1 + \dots + \mathbb{C}x_4$ . Complex conjugation on  $V$  is defined by:  $\sum a_i x_i = \sum \bar{a}_i x_i$ . Let  $\ell$  define a 2-dimensional subspace  $W \subseteq V$  and  $\bar{\ell}$  define the subspace  $\bar{W} \subseteq V$ . Choose  $\{x_j, x_k\}$  as a basis of  $V/W$ .

For  $V/\bar{W}$  we choose  $\bar{x}_j, \bar{x}_k$  as representatives of a basis. This defines the map  $f : \ell \times \bar{\ell} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . The real structure considered on  $\mathbb{P}^1 \times \mathbb{P}^1$  is the one induced by  $f$ . In particular, for  $p = (a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$  one has  $\bar{p} = (\bar{b}, \bar{a})$ .

In order to construct the map in Step (6) one must define complex conjugation on the forms of bi-degree  $(1, 1)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ . This is done as follows. Now, a map  $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$  is constructed as follows. Let  $X_1, X_2$  and  $X'_1, X'_2$  denote the homogeneous coordinates of  $\mathbb{P}^1$  and  $\mathbb{P}^1$ . Complex conjugation on the forms of bi-degree  $(1, 1)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  (with the above real structure) is given by:  $\sum \alpha_{ij} X_i X'_j = \sum \bar{\alpha}_{ij} X'_i X_j$ . A basis  $\{f_1, f_2, f_3\}$  for the 3-dimensional real space of the forms of bi-degree  $(1, 1)$  vanishing at  $p_1$ , and invariant under the complex conjugation as defined here, defines the map  $\Phi_2$ .

**Remark 4.1.** Note that the above algorithm fails for surfaces with 13 real tritangent planes already in the very first step: there exists no pair of skew conjugate lines on the surface.

## 5. Explicit examples

Next, we apply the algorithm in order to calculate explicit parameterizations for two surfaces, namely the Clebsch diagonal surface and the Fermat cubic surface (see Fig. 2).

### 5.1. The Clebsch diagonal surface

The Clebsch diagonal surface, discovered by Clebsch in 1871, is a nonsingular cubic surface  $S$  with the property that all its 27 lines are defined over  $\mathbb{R}$ . It is given by the following equations in  $\mathbb{P}^4$ :

$$S = \begin{cases} x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \\ x_0 + x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

An equation for  $S$  in  $\mathbb{P}^3$  is  $x_0^3 + x_1^3 + x_2^3 + x_3^3 + (-x_0 - x_1 - x_2 - x_3)^3 = 0$ . The 27 real lines on  $S$  can be given in two groups as 15 lines defined over  $\mathbb{Q}$  of the form:

$$l_{i,jk} = \begin{cases} x_i = 0, \\ x_j + x_k = 0, \end{cases}$$

where  $i, j, k \in \{0, 1, 2, 3, 4\}$  and 12 lines defined over  $\mathbb{Q}(\sqrt{5})$  and not over  $\mathbb{Q}$  which can be written as:

$$\Delta_{(j,k,m,n)} = \begin{cases} x_j + \tau x_k + x_m = 0, \\ x_k + \tau x_j + x_n = 0, \\ \tau x_j + \tau x_k - x_4 = 0 \end{cases}$$

with  $\{j, k, m, n\} = \{0, 1, 2, 3\}$ ,  $j < k$  and  $\tau = \frac{1+\sqrt{5}}{2}$ .

Consider the following two skew lines on  $S$

$$\ell_1 = \begin{cases} x_0 = 0, \\ x_1 + x_2 = 0. \end{cases} \quad \ell_2 = \begin{cases} x_4 = 0, \\ x_0 + x_2 = 0. \end{cases}$$

Apply the algorithm to the two lines  $\{\ell_1, \ell_2\}$  as follows.

- Recall that the morphism  $\Phi_1(p) = (p_1, p_2) \in \ell_1 \times \ell_2$ , where  $p_1$  and  $p_2$  are the intersection points of  $\ell$  with  $\ell_1$  and  $\ell_2$  respectively, and  $\ell$  is the unique line passing through  $p$  and intersecting both the lines. Hence, the points  $p_1 = (0 : -a - c : a + c : a + b + c + d)$  and  $p_2 = (-a : -a - b - c : a : a + b + c)$  are the intersection points  $H_2 \cap \ell_1$  and  $H_1 \cap \ell_2$  respectively, where  $H_i$  are the planes passing through  $\ell_i$  and  $p = (a : b : c : d)$  for  $i = 1, 2$ .
- Consider  $V = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3 + \mathbb{C}x_4$  in  $\mathbb{C}[x_1, \dots, x_4]$ , and  $W \subseteq V$  the subspace given by  $\{h \in V \mid h(\ell_1) = 0\}$  and  $W'$  the subspace  $\{h \in V \mid h(\ell_2) = 0\}$ . Consider the bases  $\{\beta_1, \beta_2\} = \{x_2, x_3\}$  and  $\{\tilde{\beta}_1, \tilde{\beta}_2\} = \{x'_2, x'_3\}$  of the spaces  $V/W$  and  $V/W'$  respectively. Define  $f : \ell_1 \times \ell_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  as  $((x_0 : x_1 : x_2 : x_3), (x'_0 : x'_1 : x'_2 : x'_3)) \mapsto ((x_2 : x_3), (x'_2 : x'_3))$ . The map  $f \circ \Phi_1 : S \subseteq \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is therefore given by  $(a : b : c : d) \mapsto ((a + c : a + b + c + d), (a : a + b + c))$ .
- Find the 5 lines  $m_1, \dots, m_5$  on  $S$  that intersect both  $\ell_1$  and  $\ell_2$  (see Lemma 1.2).
- By the definition of  $\Phi_1$ , the five lines  $m_i$  are contract to five points:
  - $p_1 = f \circ \Phi_1(m_1) = ((1 : 0), (-1 : 1))$ ,
  - $p_2 = f \circ \Phi_1(m_2) = ((1 : 1), (0 : 1))$ ,
  - $p_3 = f \circ \Phi_1(m_3) = ((0 : 1), (1 : 0))$ ,
  - $p_4 = f \circ \Phi_1(m_4) = ((\bar{\tau} : 1), (-\tau : 1))$ ,
  - $p_5 = f \circ \Phi_1(m_5) = ((\tau : 1), (-\bar{\tau} : 1))$ ,
 where  $\bar{\tau} = \frac{1-\sqrt{5}}{2}$ . Consider any of the points  $p_i$  (since they are all real), for instance, the point  $p_2 = \Phi_1(m_2) = ((1 : 1), (0 : 1))$ .
- Consider a basis for the subspace of the forms of bi-degree  $(1, 1)$  in  $\mathbb{C}[X_2, X_3, X'_2, X'_3]$  that vanish at the point  $((1 : 1), (0 : 1))$ , for example the basis  $\mathcal{B} = \{X_2X'_3 - X_3X'_2, X_3X'_2, X_2X'_2\}$ . This basis defines the map  $\Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$  as  $((x_2 : x_3), (x'_2 : x'_3)) \mapsto (x_2x'_3 - x_3x'_2 : x_3x'_2 : x_2x'_2)$ .
- The complete morphism  $\Phi = \Phi_2 \circ f \circ \Phi_1 : S \subseteq \mathbb{P}^3 \rightarrow \mathbb{P}^2$  is given by  $\Phi((a : b : c : d)) = ((-b - d)(a + b + c) : a(a + b + c + d) : a(a + c))$ . Its inverse is the desired parametrization.

Note that four of the six base points in  $\mathbb{P}^2$  are the images  $\Phi(m_1)$ ,  $\Phi(m_3)$ ,  $\Phi(m_4)$ ,  $\Phi(m_5)$ . The remaining two are the images of the lines  $r_1 = \{(1 : 1)\} \times \mathbb{P}^1$  and  $r_2 = \mathbb{P}^1 \times \{(0 : 1)\}$  under  $\Phi_2$ . That is, the six points are:

$$(1 : 0 : 1), \quad (0 : 1 : 0), \quad (-\tau : -\tau : 1), \quad (-\bar{\tau} : -\bar{\tau} : 1), \quad (1 : 0 : 0), \quad (0 : 1 : 1).$$

The six lines on  $S$  that are contract to the six points are the lines  $m_1, m_3, m_4, m_5, \Phi_1^{-1}(r_1)$  and  $\Phi_1^{-1}(r_2)$ . Finally, the inverse map  $\psi = \Phi^{-1} : \mathbb{P}^2 \dashrightarrow S \subseteq \mathbb{P}^3$  which maps  $(x : y : z)$  to  $((-z + y)(-z^2 + xz + xy) : -z^3 + xz^2 + yz^2 - x^2y : x(z^2 - xz - y^2) : z(-xz + x^2 + yz - y^2))$  is a parametrization of  $S$  given by polynomials of degree 3.

## 5.2. The Fermat cubic surface

The Fermat cubic  $S$  is the nonsingular cubic surface in  $\mathbb{P}^3$  defined by the polynomial  $x_0^3 + x_1^3 + x_2^3 + x_3^3$ . The 27 lines on the Fermat can be given in three groups of 9 lines

$$\begin{cases} x_0 + \omega x_1 = 0, \\ x_2 + \omega' x_3 = 0, \end{cases} \quad \begin{cases} x_0 + \omega x_2 = 0, \\ x_1 + \omega' x_3 = 0, \end{cases} \quad \begin{cases} x_0 + \omega x_3 = 0, \\ x_1 + \omega' x_2 = 0, \end{cases}$$

where  $\omega$  and  $\omega'$  are cubic roots of unity (not necessarily primitive).

The Fermat cubic does not contain a skew pair of real lines. Hence, we consider a pair of skew complex conjugate lines and apply the algorithm as described in Section 4. Now fix a primitive third root of unity  $\omega$ . Consider the following pair of skew conjugate lines:

$$\ell = \begin{cases} x_0 + \omega x_1 = 0, \\ x_2 + \omega x_3 = 0, \end{cases} \quad \bar{\ell} = \begin{cases} x_0 + \omega^2 x_1 = 0, \\ x_2 + \omega^2 x_3 = 0. \end{cases}$$

- As in the previous example, we construct the map  $\Phi_1(p) = (p_1, p_2) \in \ell \times \bar{\ell}$  where  $p_1 = (-a\omega - b : a + b\omega^2 : -\omega(c + d\omega^2) : c + d\omega^2)$  and  $p_2 = (-a\omega^2 - b : a + b\omega : -\omega^2(c + d\omega) : c + d\omega)$ .
- Let  $V_{\mathbb{R}} = \mathbb{R}x_1 + \dots + \mathbb{R}x_4$  and  $V = \mathbb{C}x_1 + \dots + \mathbb{C}x_4$ . Let  $\ell$  define a 2-dimensional subspace  $W \subseteq V$  and  $\bar{\ell}$  define the subspace  $\bar{W} \subseteq V$ . Consider the bases  $\{x_1, x_3\}$  and  $\{x'_1, x'_3\}$  of the spaces  $V/W$  and  $V/\bar{W}$  respectively. Define  $f : \ell \times \bar{\ell} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  as  $((x_0 : x_1 : x_2 : x_3), (x'_0 : x'_1 : x'_2 : x'_3)) \mapsto ((x_1 : x_3), (x'_1 : x'_3))$ .
- The map  $f \circ \Phi_1 : S \subseteq \mathbb{A}^3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  maps  $(a : b : c : d)$  to  $((a + b\omega^2 : c + d\omega^2), (a + b\omega : c + d\omega))$ .
- As before, the five lines that intersect both the lines  $\ell$  and  $\bar{\ell}$  are computed (see Lemma 1.2). Note that at least one of them, say  $m_5$ , is real.
- By the definition of  $\Phi_1$ , the five lines  $m_1 \dots m_5$  are sent to five points:
  - (1)  $f \circ \Phi_1(m_1) = ((1 : 0), (0 : 1))$ ,
  - (2)  $f \circ \Phi_1(m_2) = ((0 : 1), (1 : 0))$ ,
  - (3)  $f \circ \Phi_1(m_3) = ((-\omega : 1), (-\omega : 1))$ ,
  - (4)  $f \circ \Phi_1(m_4) = ((-\omega^2 : 1), (-\omega^2 : 1))$ ,
  - (5)  $f \circ \Phi_1(m_5) = ((-1 : 1), (-1 : 1))$ .
 Consider the real point  $p_5 = ((-1 : 1), (-1 : 1))$ .
- After finding basis for the subspace of forms of bi-degree  $(1, 1)$  that are invariant under complex conjugation and vanishing at  $((-1 : 1), (-1 : 1))$ , one defines  $\Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$  as  $((x_1 : x_3), (x'_1 : x'_3)) \mapsto (2x_1x'_1 + x_1x'_3 + x'_1x_3 : \frac{x_1x'_3 - x'_1x_3}{2\omega + 1} : -x_1x'_1 + x_3x'_3)$ .
- The complete morphism  $\Phi_2 \circ f \circ \Phi_1 : S \subseteq \mathbb{P}^3 \rightarrow \mathbb{P}^2$  is given by  $(a : b : c : d) \mapsto (2a^2 - a(2b - 2c + d) + b(2b - c + 2d) : ad - bc : -a^2 - b^2 + c^2 + ab - cd + d^2)$ .

**Remark 5.1.** An explicit morphism for the Fermat cubic surface, different from ours, was first presented by N. Elkies (<http://www.math.harvard.edu/~elkies/4cubes.html>).

The six base points in  $\mathbb{P}^2$  for the Fermat cubic that correspond to this contraction are

$$(\omega : 1 : 0), \quad (\omega^2 : 1 : 0), \quad (2 : 0 : \omega), \quad (2 : 0 : \omega^2), \quad (\omega : 1 : -\omega), \quad (\omega^2 : 1 : -\omega^2),$$

which are  $\Phi(m_1)$ ,  $\Phi(m_2)$ ,  $\Phi(m_3)$ ,  $\Phi(m_4)$ ,  $\Phi_2(r_1)$  and  $\Phi_2(r_2)$ . The six lines that are contracted to these six points are the four lines  $m_1, m_2, m_3, m_4$  and the lines  $\Phi_1^{-1}(r_1)$  and  $\Phi_1^{-1}(r_2)$ , where  $r_1 = \{(-1 : 1)\} \times \mathbb{P}^1$  and  $r_2 = \mathbb{P}^1 \times \{(-1 : 1)\}$ .

The map  $\Psi = \Phi^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$  maps  $(x : y : z)$  to:

$$\begin{aligned} &(-x^3 - 2x^2z + 3x^2y + 12yxz - 3xy^2 - 4xz^2 + 6y^2z + 12yz^2 + 9y^3 : \\ &x^3 + 2x^2z + 3x^2y + 12yxz + 3xy^2 + 4xz^2 - 6y^2z + 12yz^2 + 9y^3 : \\ &-8z^3 - 8xz^2 - 9y^3 - x^3 - 3x^2y - 3xy^2 - 4x^2z - 12y^2z : \\ &8z^3 + 8xz^2 - 9y^3 + x^3 - 3x^2y + 3xy^2 + 4x^2z + 12y^2z) \end{aligned}$$

which gives a parametrization of the surface by cubic polynomials.

**Remark 5.2.** Leonhard Euler was the first one to give all rational solutions of  $\sum_{i=0}^3 x_i^3 = 0$ . His parameterization is given by the following polynomials of degree 4:

$$\begin{aligned} x_0 &= c^4 - c(a - 3b)(a^2 + 3b^2), \\ x_1 &= c(a + 3b)(a^2 + 3b^2) - c^4, \end{aligned}$$



$$x_2 = (a^2 + 3b^2)^2 - c^3(a + 3b),$$

$$x_3 = (a - 3b)c^3 - (a^2 + 3b^2)^2.$$

The details of his method can be found in Dickson (1919, pp. 552–554).

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